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On the angular motion of a freely falling human or animal body

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SUMMARY

From the earliest times it is known that a cat always falls on its feet even when dropped upside down. This behaviour, clearly showing that in this relation it is not correct to consider an animal body as a rigid one, was qualitatively understood by Lecornu in 1894. Some decades later Rademaker and ter Braak proposed a simple model of the animal body, yet retaining its capacity of turning freely. In this paper after a short historical survey, the main contribution of the first author to it, a rational analysis of the model is given. This yields quantitative data on the dynamic and kinematical behaviour of the human and the animal body during the free fall in the air, and also while floating in water.

1. Introduction

The proverbial ability of a cat to fall on its feet even when dropped upside down, has been the subject of a number of mutually controversial theories for a long time. Led by an, as we now know, improper approximation of the body in considering it as a rigid one, some physicists have denied that a falling cat could really turn its body as a whole, unless it communicates an initial angular velocity to itself at the time of take-off. The inference was arrived at on the basis of the law of conservation of the angular momentum about the cat's centre of gravity. Some investigators have realized that it is not allowable in this connection to treat the animal body as a rigid one. Consequently, they have proposed to take account of the internal degrees of freedom in relation to its kinematical behaviour. In doing so, however, there is more than one way in which one can explain the rotation of a cat. As we will see in the sequel, two mutually competing theories have been developed. One of the two, in our opinion the less plausible one, has become known in the literature on sport and, in particular, on (competition) diving and athletics. The second theory underlies the calculations of the present paper which follow in the nex section.

We proceed with a short historical survey. The first detailed discussion we have come across, was in the French Académie des Sciences in 1894 when Marey [1], a physiologist, showed a series of photographs of a cat turning. In the same year five papers [2]-[6] commented upon Marey's experimental results. Guyou [2] gave an explanation compatible with the law of conser-

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vation of angular momentum about the centre of gravity. He assumes an alternating torsion of the upper part of the body with regard to the abdomen. If during this process the cat varies the moment of inertia about the longitudinal axis of each part in a particular way, e.g. by adjusting the position of its fore- and hind-legs, he can indeed accomplish a finite overall rotation of its body. Although Guyou's analysis, for the most part also contained in [3]-[5], is reconcilable with basic laws, there are no appreciable experimental results backing it. The paper of Chaston [8], who conducted some experiments on the Guyou motion through the use of a turning table, probably has attributed to a dissemination of it in sporting circles [15], otherwise to little purpose.

Lecornu [6] was the first who gave a rational explanation. According to him each body in the form of a bent cylinder can put itself into rotation about a straight line connecting two points of the bent axis of the cylinder in the following way. Each cross section of the bent cylinder perpendicular to the bent axis is made to rotate, e.g. by alternately contracting and expanding longitudinal muscles. In doing so even a snake should be able to turn round in the air. Probably due to the condensed form and abstract style of [6], Lecornu's idea remained virtually unknown outside a small circle of physicists and physiologists. By-passing a paper of Magnus [7], in which the emphasis is laid on the physiological aspects of the problem, we mention a paper of Rademaker and ter Braak [9]. They performed a new series of experiments with a falling cat and, moreover, conceived a simplified model of the animal body. Instead of considering Lecornu's model in the form of a bent cylinder, they refer their analysis to a mechanical system consisting of two rigid circular cylinders which represent the upper and the lower part of the animal body. Applying basic laws of mechanics Rademaker and ter Braak were able to calculate some kinematical properties of system. Their analysis, however, is involved and, as a consequence, it is difficult to follow. A comprehensive series of tests with a falling cat and a diver has been carried out by McDonald [10]-[13]. As a result of these carefully implemented experiments it can be concluded that the mechanism proposed by Lecornu and by Rademaker and ter Braak, respectively, underlies the possibility of turning a human or an animal body during the free fall in the air and also while floating in water. For more historical details we refer to Gerritsen [14].

The purpose of this paper is to reconsider the model of Rademaker and ter Braak [9] and to give a rational analysis of its dynamic and kinematical behaviour. For a detailed description of the model we refer to the next section, the calculations of which are in keeping with the simplicity of the basic mechanism discussed above. Moreover, it appears possible to derive optimal conditions with respect to turning with the least effort.

2. Mathematical formulation

We consider a pair of identical, circular cylinders I and II, which are assumed to be homogeneous and rigid (Fig. 1).

The centres of gravity are denoted by C' and C'', respectively, the radius is denoted by r and the height by ℓ. The axes of the cylinders are permanently coupled to each other in a point 0 and the lower rim of the upper cilinder I is supposed to be contiguous to the upper rim of the lower cylinder II in some point A, say. Further it is assumed that I and II are connected to each other,

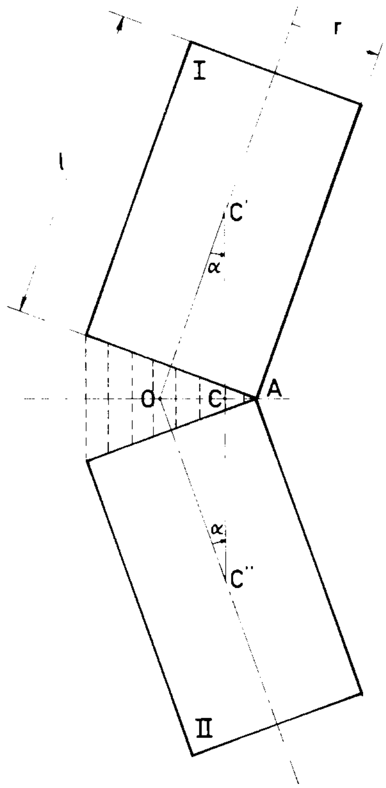

Figure 1. Geometry of the model

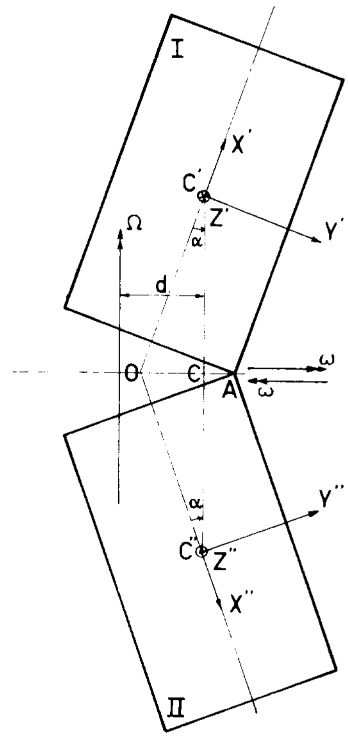

Figure 2. Angular velocities of the model

e.g. by means of strings stretching between the rims mentioned above. It is clear that the cylinders are meant as a, very crude, approximation of the upper and lower part of the body and that the strings represent the muscles. In what follows we will assume that the (massless) strings can expand and contract in an arbitrarily prescribed manner, rendering the system rheonomic. The latter is considered to move under the influence of gravity and of the internal forces exerted by the strings. Since the latter forces are in equilibrium, and as we are interested only in the rotations of the system parts, we leave the translation out of consideration and refer the system to a frame of reference moving with the downward acceleration g of gravity. In this way we may consider the centre of gravity C of the whole system as fixed in space. Specification of the inclination of the system in space is immaterial to our calculations.

We proceed to describe the motion of the two parts of the body. If as a result of muscular activity the cylinders I and II, initially at rest, should acquire an angular velocity ω and $-\omega$, respectively, about the line OA, as indicated in Fig. 2, then a velocity in a direction perpendicular to the plane OAC' would be communicated to the centres of gravity C' and C''. Since there is no resultant force, an angular velocity Ω will be induced compensating for this velocity. In view of symmetry Ω will rotate the whole model about an axis parallel with the line $C'C''$ at a distance d, so that (Fig. 2)

$$\omega\left(\frac{\ell}{2}\cos\alpha + r\sin\alpha\right) - \Omega d = 0, \tag{2.1}$$

in which α is the angle of inclination of the cylinder, as shown in Fig. 1. We assume that ω and Ω are constant so that the meridional plane OAC' executes a stationary rotation about the line $C'C''$. The ensuing calculations are referred to the, likewise rotating, cartesian coordinate systems X', Y', Z' and X'', Y'', Z'' with origins C' and C'', respectively. The axes X', Y', X'' and Y'' remain in the meridional plane (Fig. 2).

Considering first cylinder I, we find for the components of the angular velocity ω' of I with respect to the reference system X', Y', Z'

$$\begin{aligned} \omega'_{x'} &= \Omega\cos\alpha + \omega\sin\alpha, \\ \omega'_{y'} &= -\Omega\sin\alpha + \omega\cos\alpha, \\ \omega'_{z'} &= 0. \end{aligned} \tag{2.2}$$

From this the components of the vector $\mathbf{D}'$ of angular momentum about C' appear to be

$$\begin{aligned} D'_{x'} &= I_x(\Omega\cos\alpha + \omega\sin\alpha), \\ D'_{y'} &= I_y(-\Omega\sin\alpha + \omega\cos\alpha), \\ D'_{z'} &= 0, \end{aligned} \tag{2.3}$$

in which I_x and I_y denote the moment of inertia about the X'- and Y'-axis, respectively. Applying similar calculations to II or, shorter, from symmetry, we find that the vector $\mathbf{D}$ of the moment of momentum of the total system with regard to C is directed along $C'C''$ and its magnitude is

$$D = 2\{I_x(\Omega\cos\alpha + \omega\sin\alpha)\cos\alpha - I_y(-\Omega\sin\alpha + \omega\cos\alpha)\sin\alpha\}. \tag{2.4}$$

Since the system started from rest, and as D obviously has to be conserved, we find that (2.4) must vanish. This yields

$$\Omega = \frac{(I_y - I_x)\omega\sin\alpha\cos\alpha}{I_x\cos^2\alpha + I_y\sin^2\alpha}. \tag{2.5}$$

From this it is apparent that an angular velocity ω controlled by muscle, can indeed give rise to an overall rotational velocity Ω. The sign of Ω is seen to depend on the ratio I_y/I_x. Applying the model to the human or animal body, we have $I_y > I_x$, so that $\Omega > 0$. (If $I_y = I_x$, then Ω vanishes and we have to reconsider our model. In this case there exists no finite value of d and, in order to satisfy (2.1), we introduce a uniform linear velocity replacing Ωd in (2.1).) It is interesting to compare the material velocity v of A with the velocity w with which the point of contact seemingly moves in space along a circle having its centre at C. Choosing the positive Z''-axis as the positive direction of these velocities, we find using (2.5)

$$\nu = -\omega'_x{}'r - \omega'_y{}'\frac{\ell}{2} = \frac{-\left(I_y r \sin\alpha + I_x \frac{\ell}{2}\cos\alpha\right)\omega}{I_x\cos^2\alpha + I_y \sin^2\alpha}, \tag{2.6}$$

and

$$w = (\Omega - \omega \cot\alpha)\left(\frac{\ell}{2}\sin\alpha - r\cos\alpha\right) = \frac{-I_x\cos\alpha\left(\frac{\ell}{2}\sin\alpha - r\cos\alpha\right)\omega}{(I_x\cos^2\alpha + I_y\sin^2\alpha)\sin\alpha}. \tag{2.7}$$

We see that for $0 \leq \alpha \leq \pi/2$ the signs of ν and ω are always different. The sign of w, however, is dependent upon the ratio of some geometrical quantities. If one estimates the following values for a human body: $\ell \sim 0{,}85$ m, $r \sim 0{,}15$ m and $\alpha \sim 30°$, then the nominator of (2.7) becomes negative.

Finally we note that, instead of proceeding from ω and Ω, we could have started the calculations using different components of the angular velocity just as well. For instance, if we use the components ω_1, ω_2 of the angular velocities of the cylinders, so that ω_1 is directed along $C'C''$ and ω_2 along the axis of the pertaining cylinder, then we arrive at

$$\omega_1 = -\frac{I_x\cos\alpha}{I_x\cos^2\alpha + I_y\sin^2\alpha}\,\omega_2. \tag{2.8}$$

As was to be expected ω_1 and ω_2 have different signs for $|\alpha| < \pi/2$.
From (2.8) and

$$\Omega = \omega_1 + \omega_2 \cos\alpha \tag{2.9}$$

we find

$$\Omega = \frac{(I_y - I_x)\sin^2\alpha\cos\alpha\,\omega_2}{I_x\cos^2\alpha + I_y\sin^2\alpha}, \tag{2.10}$$

which result is in accordance with (2.5) since $\omega = \omega_2 \sin\alpha$. We note that in both cases (2.5) and (2.10) $\Omega = 0$ for $\alpha = 0$ and $\alpha = \pi/2$. This gives rise to consider optimal turning conditions.

3. Optimal turning conditions

It does not seem unreasonable to consider ω to be controlled by muscle in a manner not depending on the angle α of inclination. If so, we can calculate the maximal value of Ω in the interval $0 < \alpha < \pi/2$ by differentiating (2.5) with respect to α. Writing $c = (I_y/I_x) - 1$, we find

$$\frac{\Omega_{\max}}{\omega} = \frac{c}{2\sqrt{1+c}} \tag{3.1}$$

occurring at

$$\alpha_{cr} = \tfrac{1}{2} \arccos \left(\frac{c}{2+c}\right). \tag{3.2}$$

Some numerical values derived from (3.1) and (3.2) have been collected in Table 1 for $c > 0$.

Table 1

c	Ω_{max}/ω	α_{cr}
0	0	45°
2	0,58	30°
5	1,02	22,2°
10	1,51	16,8°
100	4,98	5,7°
∞	∞	0

On the other hand, if we consider the component ω_2 of the angular velocity, introduced at the end of the preceding section, as a primary quantity determined by muscular activity, then we have to look for the maximal value of the function (2.10).
In this case we arrive at the following critical value of the angle α

$$\alpha_{cr} = \arcsin \left\{\frac{-3+\sqrt{9+8c}}{2c}\right\}^{1/2}, \tag{3.3}$$

leading to a complicated expression for Ω_{max}, which we omit here. Table 2 contains some numerical values applying to this case.

Table 2

c	$\dfrac{\Omega_{max}}{\omega_2}$	α_{cr}
0	0	54,7°
2	0,35	45 °
5	0,52	39,2°
10	0,63	34,6°
100	0,87	20,9°
∞	1	0

4. Concluding remarks

In order to apply the above numerical results to the human or animal body, we have to estimate the moments of inertia I_x and I_y. If we retain the rude approximation that the cylinders are homogeneous, then

$$c = \frac{1}{6}\left(\frac{\ell}{r}\right)^2 - \frac{1}{2}. \qquad (4.1)$$

Using the estimates of Section 3 we arrive at a value $c \sim 5$ for the human body. For animals like cats we probably have to apply lower values, say $c \sim 2$. From the two tables it appears: the slender a body the greater the induced angular velocity. The assumption underlying Table 2 is seen to yield somewhat larger values for the most favourable value α_{cr} of the angle of inclination than those contained in Table 1. However, the range of values calculated agrees reasonably with what has been found experimentally. In [11] and [12] a value $\alpha_{cr} \sim 45°$ is given in relation to cats and in [12] we find $\alpha_{cr} \sim 10° - 15°$ for a man. From [13], however, we estimate larger values for the latter case, say $\alpha_{cr} \sim 20°$. The empirical values shown by men thus seem to conform to Table 1, while the behaviour of a cat is more in accordance with Table 2.

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